

On fitting the Pareto-Levy distribution to stock market index data: selecting a suitable cutoff value.

H.F. Coronel-Brizio ^a and A.R. Hernandez-Montoya ^b

Facultad de Física e Inteligencia Artificial. Universidad Veracruzana, Apdo. Postal 475. Xalapa, Veracruz. México.

Received: date / Revised version: date

Abstract. The so-called Pareto-Levy or power-law distribution has been successfully used as a model to describe probabilities associated to extreme variations of worldwide stock markets indexes data and it has the form $Pr(X > x) \simeq x^{-\alpha}$ for $\gamma < x < \infty$. The selection of the threshold parameter γ from empirical data and consequently, the determination of the exponent α , is often is done by using a simple graphical method based on a log-log scale, where a power-law probability plot shows a straight line with slope equal to the exponent of the power-law distribution. This procedure can be considered subjective, particularly with regard to the choice of the threshold or cutoff parameter γ . In this work is presented a more objective procedure, based on a statistical measure of discrepancy between the empirical and the Pareto-Levy distribution. The technique is illustrated for data sets from the New York Stock Exchange Index and the Mexican Stock Market Index (IPC).

Key words. Econophysics, power-law, returns distribution, fit, empirical distribution function.

PACS. 01.75.+m Science and Society - 02.50.-r Probability theory, stochastic processes and statistics - 02.50.Ng Distribution Theory and Monte Carlo studies - 89.65.Gh Economics; econophysics, financial markets, business and management - 89.90.+n Other areas of general interest to physicists

1 Introduction

The Power-law distribution is present in a great scope of physical (phase transitions, nonlinear dynamics and disordered systems) [1,2], financial (stocks prices and indexes variations, volumes, volatility decay distributions) [3,4,5,6,7,8,9,10], and other kinds of social phenomena (the World Wide Web and Internet router links, sexual contact networks, growth of cities, reference networks in scientific journals, University entrance examinations, and traffic penalties distributions) [11,12,13,14,15,16,17]. All these systems share the property of complexity and are driven by collective mechanisms of which signature is the power-law distribution.

Studying variations of financial data is important in order to understand the stochastic process that drives them and also for practical purposes related to investment and risk management.

In the analysis of stock market indexes variations, many observables are used[18]. In this work we have chosen the returns series. Briefly reviewing the definition of series, if in general, $X(t)$ denotes the value of a particular index at time t , its return series is defined as $S(t) = \log X(t+\Delta t) - \log X(t)$; that is, as the logarithmic changes in the values

of the index for a certain interval of time Δt , which can be studied within a few seconds to a many days range.

It has been reported in several empirical studies, that in order to describe the probabilities of extreme returns variations, the Pareto-Levy distribution is an useful model to compute probabilities. However, when fitting the power-law to empirical data, the choice of the threshold or cut-off parameter of the tail distribution does not seem to follow an objective procedure. Usually, its fitted value is obtained by judging the degree of linearity in a log-log plot involving the empirical and theoretical distributions, even more, recently, well founded studies have criticize the reliability of this geometrical method [19,20] .

Independently of these studies just cited above, we propose in this work, a formal procedure to improve the quality of the log-log plot fit based on measures of discrepancy between the empirical and theoretical distribution functions. In order to illustrate our technique, we study the daily returns distribution of both, an emergent and a well developed stock markets: the Mexican stock market index IPC ¹ and the American DJIA ². Figures 1 and 2 show

^a e-mail: hcoronel@uv.mx

^b e-mail: alhernandez@uv.mx

¹ Índice de Precios y Cotizaciones, which means Prices and Quotations index

² Dow Jones Industrial Average Index

daily returns distributions respectively for these financial markets.

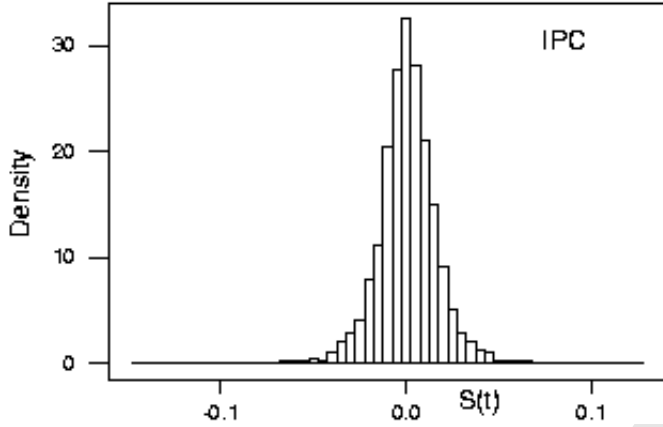


Fig. 1. Density histogram for daily logarithmic differences of the Mexican IPC index, from April 19 1990 to September 17 2004

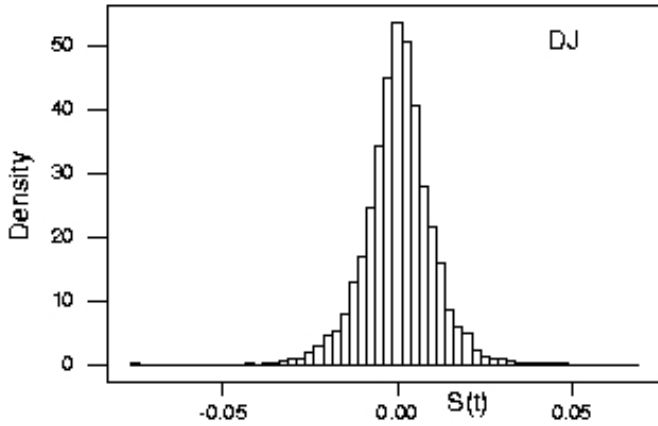


Fig. 2. Density histogram for daily logarithmic differences of the Dow Jones index, from April 19 1990 to September 17 2004

2 Choosing the value of the threshold parameter

Let $S(s)$ be an absolutely continuous random variable and let us assume that there exists a value γ , such that, for $s > \gamma$, S follows a Pareto-Levy distribution with parameters $\alpha > 0$ and γ ; that is:

When $s > \gamma$,

$$F_0(s) = 1 - \left(\frac{\gamma}{s}\right)^\alpha \quad (1)$$

If the value γ was known, we could compute a measure of fit of the Pareto-levy model using, for example, quantities within the so-called quadratic statistics:

$$Q_n = \int_{-\infty}^{\infty} [F_n - F_0]^2 \psi dF_0$$

where F_n denotes the empirical distribution function and ψ is a weighting function; for example:

- For $\psi(z) = 1$, Q_n becomes Watson's W^2 statistic.
- For $\psi(z) = \{F_0(z)[1 - F_0(z)]\}^{-1}$ Q_n it is the well known Anderson-Darling A^2 statistic.

Focusing in the former, if we denote by $A^2(\gamma)$ the computed value of the A^2 statistic for a given value γ , the fitted value of γ can be chosen as the value which minimizes $A^2(\gamma)$. In the next section the computational details will be described; however, for a more detailed treatment of statistics based on the empirical distribution function, the interested reader is referred to [21].

3 Computing formulas

Let $s_{(1)} \leq s_{(2)} \leq \dots \leq s_{(n(\gamma))}$ denote the ordered values of the observed series $s(t_1), s(t_2), \dots, s(t_{n(\gamma)})$, where $s(t_i) \geq \gamma$ for $i = 1, \dots, n(\gamma)$ and $n(\gamma)$ denotes the number of remaining observations in the sample which are greater than or equal a given admissible value of γ . Then the procedure steps of our method, can be enumerated as follows:

1. Estimate the shape parameter $\alpha(\gamma)$ by

$$\alpha(\gamma) = \left[\frac{1}{n(\gamma)} \sum_{i=1}^{n(\gamma)} \log \left(\frac{s_i}{\gamma} \right) \right]^{-1}$$

2. For $i = 1, \dots, n(\gamma)$, compute the quantities

$$z_{(i)} = 1 - \left(\frac{\gamma}{s_{(i)}} \right)^{\alpha(\gamma)}$$

3. Compute the value of the Anderson-Darling statistic using

$$A^2(\gamma) = -n(\gamma) - \frac{1}{n(\gamma)} \sum_{i=1}^{n(\gamma)} (2i-1) [\log z_{(i)} + \log \{1 - z_{(n-i+1)}\}]$$

Starting with, say, $\gamma_1 = s_{(1)}$ in the complete sample, a sequence $\gamma_1, \dots, \gamma_r$ can be constructed, to a desired accuracy, to produce the sequence of values $A^2(\gamma_1), \dots, A^2(\gamma_r)$. A plot of γ_r versus $A^2(\gamma_r)$ will be useful, as it will be shown in the next section, for finding the value of γ which minimizes the value of A^2 .

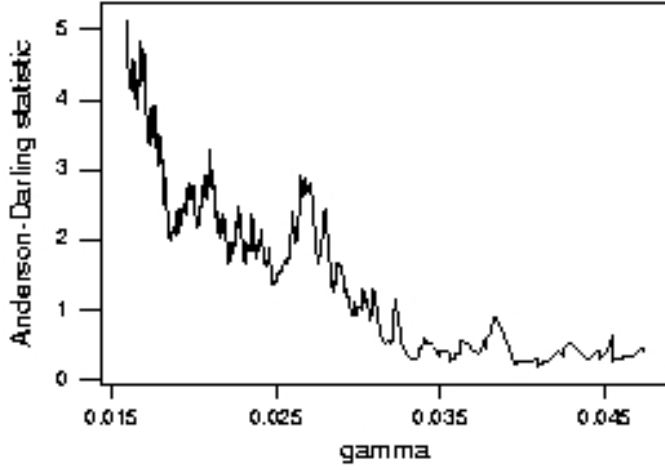


Fig. 3. Anderson-Darling A^2 statistic versus selected values of the threshold parameter γ , corresponding to the positive values of the series $S(t)$, computed from the IPC index data. The minimum value, 0.16, is attained for $\gamma = 0.0395$

4 Data Analysis

In order to illustrate the technique, two data sets were analyzed. For both data sets, the daily returns series $S(t)$ was constructed. The first one, consists of daily values of the Mexican stock market index (IPC), covering from April 19, 1990 to September 17, 2004. The $S(t)$ series which we will denote here as $S_{IPC}(t)$, had 3608 values from which 1877 positive, and 1723 negative values, were used for the analysis. The second data set consists of daily values of the Dow Jones index recorded from April 19, 1990 to September 17, 2004. The $S(t)$ series constructed from this data, denoted here by $S_{DJ}(t)$, had 3633 values. Here the analysis was based on 1899 and 1723 positive and negative values, respectively. Data bases for the IPC and DJIA were downloaded from [22] and [23] respectively.

Using the procedure described in the previous section, the Pareto distribution was fitted to the positive and negative tails of the distributions of $S_{IPC}(t)$ and $S_{DJ}(t)$ varying the value of the parameter γ over the ordered sample values. In each case, the Anderson-Darling statistic, was used as a goodness-of-fit criterion. It must be remarked that for the analysis of the negative tails, the values $-S(t)$ were used.

Figures 3 to 6 show the plots of A^2 versus different values of the threshold parameter γ . Using this approach, we obtained the following results:

For the IPC index data, the best possible fit for the largest values in the positive tail, is obtained for $\gamma_0 = 0.0395$, where the minimum value of A^2 is 0.16; based on the 64 largest positive observations, with an estimated value of the shape parameter $\alpha = 3.822$. For negative tail, the minimum value of A^2 was found to be 0.50, for $\gamma = 0.036$. The fitted value of α was 3.507, based on the 69 smallest observations.

For the case of the DJ index, the best positive fit gives $A^2 = 0.315$ for $\gamma = 0.0173$ with $\alpha = 3.333$; for the negative tail the results showed that the smallest value of $A^2 = 0.18$

Index	Tail	A^2	γ	α	n
IPC	Positive	0.16	0.0395	3.82	64
IPC	Negative	0.50	-0.0360	3.51	69
DJ	Positive	0.32	0.0173	3.33	158
DJ	Negative	0.18	-0.0191	3.50	124

Table 1. Best left and right tail fits, for the two analyzed series.

Index	Tail	A^2 -method	Graphical method
IPC	Positive	3.82	3.00
IPC	Negative	3.51	2.82
DJ	Positive	3.33	2.85
DJ	Negative	3.50	2.80

Table 2. Fitted values of the parameter α obtained by graphical and A^2 methods.

is attained for $\gamma = 0.0191$ with $\alpha = 3.495$; the above results were obtained from the 158 largest and the 124 smallest observations of $S_{DJ}(t)$, respectively.

Table 1 summarize these results for easy reference.

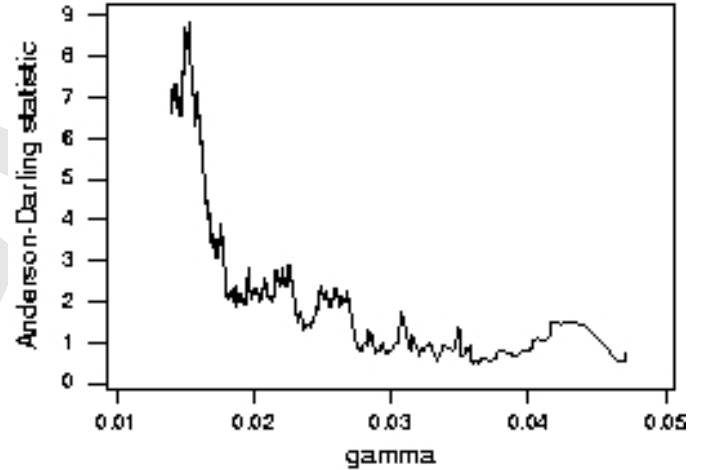


Fig. 4. Anderson-Darling A^2 statistic versus selected values of the threshold parameter γ , corresponding to the negative values of the series $S(t)$, computed from the IPC index data. The minimum value, 0.50, is attained for $\gamma = 0.036$

Table 2 suggests that our method tends to produce larger fitted values for the parameter α ; however, we can expect a much better fit using the parameter values produced by the A^2 -method for obvious reasons. As an illustration consider the fitted regression line of $\log P(s)$ on $\log s$, where $P(s) = 1 - F_n(s)$, for the positive tail of the Dow Jones $S(t)$ series shown in figure 7. The fitted value of $\alpha = 2.85$ corresponds to the slope of the regression line and it differs from our estimate $\alpha = 3.33$; figures 9 and 8 show the empirical F_n (dash), and the fitted cumulative distribution function F for each case, using $\gamma = 0.0173$. As expected, our estimates produce a better fit.

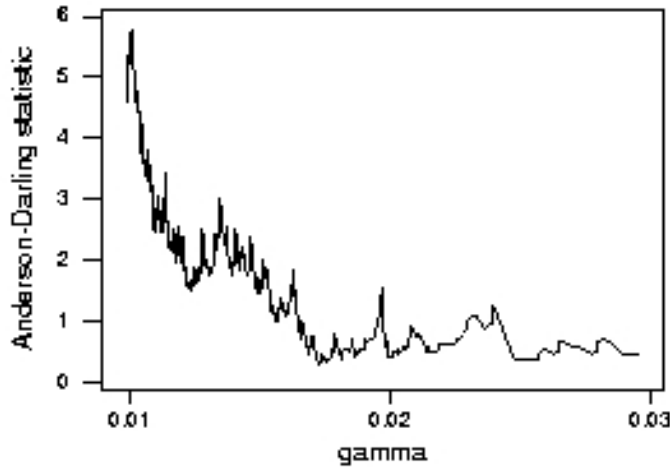


Fig. 5. Anderson-Darling A^2 statistic versus selected values of the threshold parameter γ , corresponding to the positive values of the series $S(t)$, computed from the Dow Jones index data. The minimum value, 0.32, is attained for $\gamma = 0.0173$

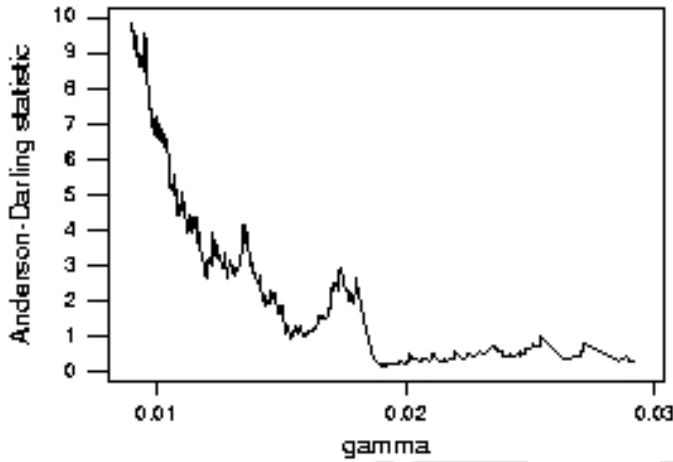


Fig. 6. Anderson-Darling A^2 statistic versus selected values of the threshold parameter γ , corresponding to the negative values of the series $S(t)$, computed from the Dow Jones index data. The minimum value, 0.18, is attained for $\gamma = 0.0191$

5 Conclusions

An objective technique for fitting the Power-Law distribution to extreme variations in stock market indexes was presented. The method is based on the use of Anderson-Darling Statistic A^2 as a measure of discrepancy between the empirical and the theoretical distribution functions, selecting as the fitted parameters, the values which minimize such a measure. The technique was illustrated for the case of the Dow Jones industrial average index and for the Mexican prices and quotations index. The results showed that this method can be used with better results than the traditional graphical method in which the value of the cutoff parameter γ is chosen subjectively.

A.R.H.M wishes to thank Conacyt-Mexico for financial support provided under Grant No. 44598-F. Also we thank useful suggestions and bibliographical support from P. Giubellino.

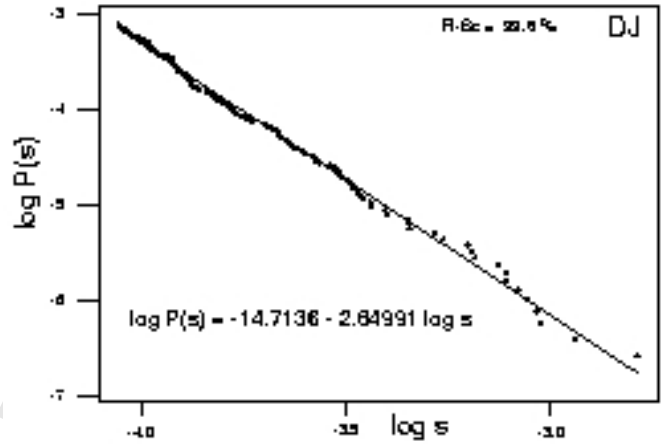


Fig. 7. Linear fit for the positive tail of the Dow Jones $S(t)$ series using $\gamma = 0.0173$. The fitted value of the shape parameter α is 2.85

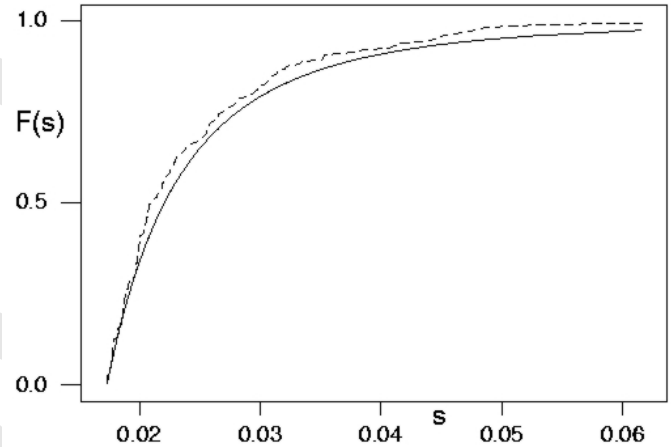


Fig. 8. Empirical (dash) and fitted (solid) cumulative distribution functions for the positive tail of the Dow Jones $S(t)$ series, using $\gamma = 0.0173$ and $\alpha = 2.85$

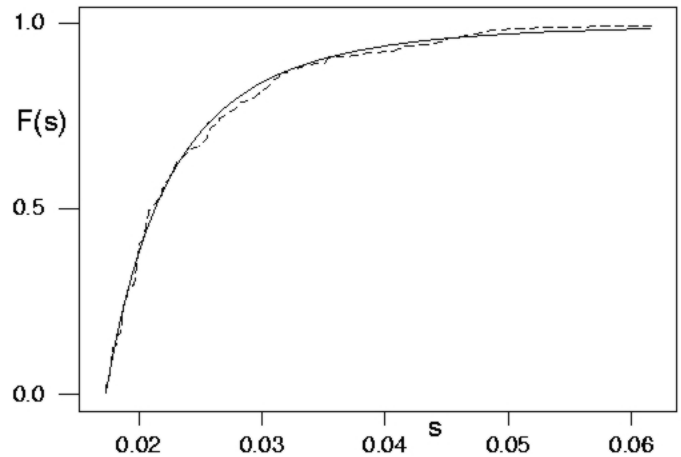


Fig. 9. Empirical (dash) and fitted (solid) cumulative distribution functions for the positive tail of the Dow Jones $S(t)$ series, using $\gamma = 0.0173$ and $\alpha = 3.33$

References

1. Goldenfeld N, Lectures on Phase Transitions and Critical Phenomena. Frontiers in Physics (1992).
2. Frisch U. Turbulence: The Legacy of A. Kolmogorov. Cambridge University Press (1997).
3. V. Plerou, P. Gopikrishnan, L. A. Amaral, M. Meyer, H. E. Stanley, Phys.Rev. E **60,6**, (1999) 6519-6529.
4. P. Gopikrishnan et al, Eur. Phys. J.B **3**, (1998) 139-140.
5. P. Gopikrishnan et al, Phys.Rev. E **60**, (1999) 5305-5316.
6. T. Lux, Applied Financial Economics **6**, (1996) 463-475
7. V.Plerou, Gopikrishnan, L. A. Amaral, H. E. Stanley, Quant. Finance **1** (2001) 262-269.
8. J. Bouchaoud, Quantitative Finance **1** (2001) 105-112.
9. P. Cizeau, Y. Liu, M. Meyer, C.-K. Peng, H. Eugene Stanley, Physica A **245** (1997) 441-445
10. Y. Liu, P. Gopikrishnan, P. Cizeau, M. Meyer, C. Peng, and H. E. Stanley Phys.Rev. E **60,2**, (1999) 1390-1400.
11. R. Albert, H. Jeong, and L. Barabasi, Review of Modern Physics **74**, 47-97 (2002).
12. R. Albert, H. Jeong and A.L. Barabasi, ., Nature **401**, 130-131 (1999).
13. F. Liljeros, C.R. Edling, L.A.N. Amaral, et al., Nature **411**, 907, (2001).
- 14.
15. S. Redner, Eur.Phys.Journ. B **4**, 131 (1998).
16. Hari M. Gupta, José R. Campanha and Fernando D. Prado. International Journal of Modern Physics C **11**, 6 1273-1279, (2000). **4**
17. Rafaella A. Nóbrega, Cricia.C. Rodegheri and Renato C. Povoas. International Journal of Modern Physics C **11**, 7 1475-1479, (2000).
18. Mantegna, R.N. and Stanley, H.E., An Introduction to Econophysics (Cambridge University Press, United Kingdom, 2000).
19. M.L. Goldstein, S.A. Morris and G.G. Yen, "Problems with fitting to the power-law distribution," European Journal of Physics B (in press).
20. Rafael Weron, . International Journal of Modern Physics C **12**, 2 209-223, (2001).
21. D'Agostino R.B. and Stephens M.A. Goodness of fit Techniques Marcel Dekker, New York, (1986).
22. IPC data base used in our analysis, is available at the Banco de México website: <http://www.banxico.org.mx>
23. www.yahoo.finance.com